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Group theoretic preliminaries to the solution of the Schrödinger equation for the manifold $SU(4)/S(U(2) \times U(2))$

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Abstract

Irreducible representations (irreps) of a compact Lie group G, of class one w.r.t. a Lie subgroup H are those that contain the identity irrep of H once in their decompositions w.r.t. H. In the case of $G = SU(4)$ and $H = S(U(2) \otimes U(2))$, the class one irreps are identified, and for them a general formula for their decomposition into irreps of H is given. This admits a useful graphical presentation. The relevance of these results to the solution of the Schrödinger equation of $SU(4)/S(U(2) \otimes U(2))$ and the state labelling difficulties encountered in implementing this solution, are discussed. For $G = SU(n+1)$ and $H = S(U(n-1) \otimes U(2))$, $n \geqslant 4$, the class one irreps of G, and hence the spectrum of the corresponding Schrödinger equation, have also been determined.

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1. Introduction

In order to describe the aims and output of this paper, we need to introduce some concepts and give some notation. First, we come to the term deficit or labelling deficit for a homogeneous space G/H, where H is a Lie subgroup of a compact Lie group G. We suppose the state labelling problem for providing a canonical completely labelled basis for the states of a generic irreducible representation (irrep for short in this paper) is solved for G and for H, as is the case for, e.g., $SU(n)$ or $SO(n)$, but not G_2. Let g and h be the numbers of labels involved in the cases of G and H, and let l_h be the rank of H. The *deficit* D of the space G/H is then defined to be

$$D = g - (h + l_h). \qquad (1)$$

The number D is the number of labels that must be adjoined to the labels provided by H to give a complete labelling of the states of a generic irrep of G. If $D = 0$ then H gives a canonical labelling, as for $G = SU(n+1)$ and $H = U(n)$, but for $D > 0$ a much less favourable situation prevails. Second, referring again to the homogeneous space G/H, and following [1], we say that an irrep of G is of *class one* w.r.t. the subgroup H of G if its decomposition into irreps of H contains the identity representation of H. If each class one irrep of G contains the identity representation of H exactly once, then H is said to be a *massive* subgroup of G. We already know that $U(n)$ [2, 3] is a massive subgroup of $SU(n+1)$ and show here that $S(U(2) \otimes U(2))$ is a massive subgroup of $SU(4)$. A result, from [1], can now be stated. If $f(g)$ is any suitable function on compact G constant on the left H-cosets of a massive subgroup H of G, then it has an expansion in terms of the basis functions of the class one irreps of G, and furthermore each class one irrep is involved exactly once in this expansion. This key result is needed because the Schrödinger equation on G/H is solved in terms of such functions $f(g)$.

In two previous papers [2, 3], we have obtained by separation of variables a complete set of solutions of the Schrödinger equation of the complex manifolds CP^2 and $CP^n = SU(n+1)/U(n)$, which are homogeneous spaces of rank 1 and deficit 0, details of the spectrum having already been known [4]. Here we wish to consider the acquisition of the same information for manifolds of higher deficit. It will soon become clear that this is a decidedly non-trivial task, increasingly so for increasing deficit. Accordingly we begin by studying a suitably simple example, namely

$$SU(4)/S(U(2) \otimes U(2)) \tag{2}$$

of rank 2 and deficit 1. Needing some convenient notation, we write

$$G(4, 2, \mathbb{C}) = SU(4)/S(U(2) \otimes U(2)) \tag{3}$$

since this is a Grassmannian manifold, as indeed is $CP^n = G(n+1, 1, \mathbb{C})$. We also write $H(2, 2, 1) = S(U(2) \otimes U(2))$, sometimes referred to loosely as $SU(2) \otimes SU(2) \otimes U(1)$. The general Grassmann manifold

$$G(n+m, m, \mathbb{C}) = SU(n+m)/S(U(n) \otimes U(m)) \tag{4}$$

for integral n, m is a Hermitian symmetric space of rank $\min(m, n)$, and deficit

$$D = (n-1)(m-1) \tag{5}$$

which is 0 for CP^n and 1 for (3). Another family

$$SO(n+2)/SO(n) \otimes SO(2) \tag{6}$$

consists of Hermitian symmetric spaces of rank 2 and deficit

$$D = s - 1 \qquad (n+2) = 2s+1 \qquad \text{and} \qquad D = s - 2 \qquad (n+2) = 2s. \tag{7}$$

The space (2), $SU(4)/S(U(2) \otimes U(2))$, is isomorphic to the $n = 4$ member of this family with D given correctly by (7). For information about spaces of the kind mentioned here, see [5].

In this paper, we confine our attention to the group theoretic preliminaries to the solution process of the Schrödinger equation on $G(4, 2, \mathbb{C})$. First, we must determine the class one irreps of $SU(4)$, i.e. irreps which are class one w.r.t. its subgroup $H(2, 2, 1)$, showing in the process that it is a massive subgroup of $SU(4)$. Using the Weyl character formula of $SU(4)$ twice, we show that these are the irreps

$$\{2r+2s, r+2s, r\} = (r, 2s, r) \qquad r, s \in \mathbb{N}_+. \tag{8}$$

It follows from the key result [1] noted above that the eigenvalues of the quadratic Casimir operator $\mathcal{C}^{(2)}$ of $SU(4)$ for these irreps give the spectrum of the Hamiltonian of $G(4, 2, \mathbb{C})$

Table 1. Irreps (r, t, r) of $SU(4)$, of class one for even t, with their dimensions and eigenvalues of $\mathcal{C}^{(2)}$.

(0, 0, 0)	(1, 0, 1)	(2, 0, 2)	(3, 0, 3)	(4, 0, 4)	(5, 0, 5)	(6, 0, 6)
1	15	84	300	825	1911	3920
0	1	$\frac{5}{2}$	$\frac{9}{2}$	7	10	$\frac{27}{2}$
(0, 1, 0)	(1, 1, 1)	(2, 1, 2)	(3, 1, 3)	(4, 1, 4)	(5, 1, 5)	(6, 1, 6)
6	64	300	960	2450	5376	10 584
$\frac{5}{8}$	$\frac{15}{8}$	$\frac{29}{8}$	$\frac{47}{8}$	$\frac{69}{8}$	$\frac{95}{8}$	$\frac{125}{8}$
(0, 2, 0)	(1, 2, 1)	(2, 2, 2)	(3, 2, 3)	(4, 2, 4)	(5, 2, 5)	(6, 2, 6)
20	175	729	2156	5200	10 935	20 825
$\frac{3}{2}$	3	5	$\frac{15}{2}$	$\frac{21}{2}$	14	18
(0, 3, 0)	(1, 3, 1)	(2, 3, 2)	(3, 3, 3)	(4, 3, 4)	(5, 3, 5)	(6, 3, 6)
50	384	1470	4096	9450	19 200	35 574
$\frac{21}{8}$	$\frac{35}{8}$	$\frac{53}{8}$	$\frac{75}{8}$	$\frac{101}{8}$	$\frac{131}{8}$	$\frac{165}{8}$
(0, 4, 0)	(1, 4, 1)	(2, 4, 2)	(3, 4, 3)	(4, 4, 4)	(5, 4, 5)	(6, 4, 6)
105	735	2640	7020	15 625	30 855	55 860
4	6	$\frac{17}{2}$	$\frac{23}{2}$	15	19	$\frac{47}{2}$
(0, 5, 0)	(1, 5, 1)	(2, 5, 2)	(3, 5, 3)	(4, 5, 4)	(5, 5, 5)	(6, 5, 6)
196	1280	4374	11 200	24 200	46 656	82 810
$\frac{45}{8}$	$\frac{63}{8}$	$\frac{85}{8}$	$\frac{111}{8}$	$\frac{141}{8}$	$\frac{175}{8}$	$\frac{213}{8}$
(0, 6, 0)	(1, 6, 1)	(2, 6, 2)	(3, 6, 3)	(4, 6, 4)	(5, 6, 5)	(6, 6, 6)
336	2079	6825	16 940	35 700	67 431	11 7649
$\frac{15}{2}$	10	13	$\frac{33}{2}$	$\frac{41}{2}$	25	30

to within an overall multiplicative constant, and $\dim(r, 2s, r)$ gives the degeneracy of the corresponding energy levels, except when accidental degeneracies occur. One example of the latter concerns $(1, 6, 1)$ and $(5, 0, 5)$, both of class one; for each of these $\mathcal{C}^{(2)}$ has eigenvalue 10, given that the normalization of $\mathcal{C}^{(2)}$ is such that $(1, 0, 1) = ad$ has eigenvalue 1. In fact, this stage of the process can be completed for $G(n+1, 2, \mathbb{C})$, the class one irreps of $SU(n+1)$ in this context being the irreps

$$(r, s, 0^{n-4}, s, r) \qquad r, s \in \mathbb{N}_+ \tag{9}$$

in highest weight notation, so that details of the spectrum of the Schrödinger equation follow.

Table 1 displays relevant class one irreps of $SU(4)$, and some others, together with their dimensions and the eigenvalues of $\mathcal{C}^{(2)}$.

Second (cf the role of the corresponding decompositions in the case of CP^n [2, 3]), one needs the complete decompositions of irreps of $(r, 2s, r)$ of $SU(4)$ into irreps $H(2, 2, 1)$ that arise when one restricts from $SU(4)$ to the subgroup. This has been accomplished here: one finds that the result can be given in a surprisingly simple form, one that furthermore can be very nicely presented diagrammatically. The result in question is (35) of section 3.1, and graphical displays for various (r, t, r), t odd as well as even, are to be found in section 5. One feature of the decompositions is the occurrence of degeneracy. Let I, I_3 and J, J_3 be labels of angular momentum type for the $SU(2)$ factors of $H(2, 2, 1)$, and let U be the eigenvalue of the $U(1)$ generator. Then irreps (I, I_3, J, J_3, U) of $H(2, 2, 1)$ occur with a degeneracy, which can be completely lifted, it turns out, by grouping the U values, for each fixed set of I, I_3, J, J_3 values, into sets $-K \leqslant U = K_3 \leqslant K$, and attaching a *formal* spin label K to the set. Thus the set of six labels (I, I_3, J, J_3, K, K_3) gives a non-degenerate labelling of the $H(2, 2, 1)$ irreps that arise in all decompositions. We stress that no recognizable $SU(2)$ transformation properties can be associated with the labels K, K_3, nor is K associated, as an eigenvalue, with any operator, but the empirical procedure described works, and the displays below all reflect it. Actually these use integral labels $x = 2I$, $y = 2J$, $z = 2K$, because the character work carried out in the derivation of decompositions more smoothly (no fractions) in this notation.

The occurrence of degeneracy is a feature encountered for $G(4, 2, \mathbb{C})$ because it is of deficit 1, being absent for manifolds such as CP^n. It is related to the assignment of a central role to $H(2, 2, 1)$ in solving the state labelling problem for states of irreps of $SU(4)$. Section 4 provides discussion of some aspects of state labelling problems for G/H of deficit $D > 0$, providing reference to and comment regarding the significant body of previous work on the topic. For G of rank l and dimension r, it is well known [6] that one needs a complete commuting set of $\frac{1}{2}(r-l)$ operators to provide, via their simultaneous eigenvalues, a complete labelling of an orthogonal basis for the states of the irreps of G. For $SU(4)$ this number is $\frac{1}{2}(15-3) = 6$. Use of a basis adapted to CP^3 features the $U(3)$, or loosely $SU(3) \otimes U(1)$, subgroup of $SU(4)$. Since $SU(3)$ provides five labels, $U(3)$ provides the required six, so that CP^3 is of deficit zero. It follows that the solution by separation of variables for the Schrödinger equation of CP^3, and likewise CP^n [3], proceeds to a successful conclusion without meeting any obstacle. This is in marked contrast to what happens for $G(4, 2, \mathbb{C})$ when one employs, as one must, an $SU(4)$ basis adapted to the subgroup $H(2, 2, 1)$: this yields only five of the six operators needed for a complete set, so that $G(4, 2, \mathbb{C})$ is of deficit 1. The degeneracy noted in the previous paragraph of course reflects this. Although the label K mentioned there empirically lifts the degeneracy, we remark again that it lacks the status of being the eigenvalue of any known operator. In fact the solution by separation of variables of the Schrödinger equation for $G(4, 2, \mathbb{C})$ is stalled at present for want of a suitably tractable sixth operator: one reaches a partial differential equation in two variables instead of a single radial equation as for CP^n. It is not obvious, on this basis, how to give a general treatment of it that reflects the now known decomposition of class one irreps of $SU(4)$ into $H(2, 2, 1)$ irreps.

Our main results then are

(i) the identification of the irreps $(r, 2s, r)$, where r and s are integers, of $SU(4)$ of class one relative to $H(2, 2, 1)$,
(ii) the explicit decomposition (35) of the irreps (r, t, r) of $SU(4)$, where r and t are integers, into $H(2, 2, 1)$ irreps and
(iii) the generalization (32) of (i) to $SU(n+1)$ and $S(U(n-1) \otimes U(2))$, for $n \geqslant 4$.

We remark on two papers which touch on matters close to the work of this paper. First, in [7] we formulated the Lagrangian dynamics of motion on $G(n+m, m, \mathbb{C})$ as a nonlinear realization of $SU(n+m)$ in which the subgroup $S(U(n) \otimes U(m)$ is realized linearly. It follows that we know the Hamiltonian of the Schrödinger equation for $G(4, 2, \mathbb{C})$ in terms of Goldstone coordinates, but not, until the state labelling problem is solved, in terms of a complete set of separation variables. Second, when $SU(6)$ was being studied as a symmetry group of the hadrons, [8] discussed the decompositions of irreps of $SU(6)$ with respect to its subgroup $H(4, 2, 1) = SU(4) \otimes SU(2) \otimes U(1)$. Using character methods different from those of this paper a good body of explicit decompositions was obtained. The method used has been applied to giving spot checks on a few of the results given here.

This paper is organized as follows. Section 2 uses the Weyl character formula for $SU(4)$ in the identification of irreps class one w.r.t. $H(2, 2, 1)$, and also gives the generalization (iii) for $SU(n+1)$. Section 3 begins by presenting, with the aid of the empirical label K, the general decomposition formula expressing characters of (r, t, r) in terms of $H(2, 2, 1)$ characters, proceeds to the easy proofs for the special cases $(0, t, 0)$ and $(r, 0, r)$, continues by sketching the means by which the latter proof can be extended to (r, t, r) for $t = 1, 2, \ldots$. In fact, the procedure can be adapted to give an inductive proof of (35), but this is not included in the text. The state labelling problem is the subject of section 4. In section 5, displays of the $H(2, 2, 1)$ decompositions of (r, t, r) are presented, as tables 2A–H, for $0 \leqslant r, t \leqslant 3$. Scrutiny of these displays indicates they contain a great deal of structure. This should suffice to make it obvious how to write down directly the display for any other case. The numbers displayed give the $z = 2K$ values at each point (x, y), and the corresponding degeneracies of irreps $\left(I = \frac{1}{2}x, I_3, J = \frac{1}{2}y, J_3, U\right)$ can be inferred from them; e.g. the $U = 0$ value at the point $(x, y) = (2, 2)$ of the irrep $(2, 3, 2)$ is three-fold degenerate.

2. Irreps of $G(4, 2, \mathbb{C})$ of class one w.r.t. $H(2, 2, 1)$

2.1. Weyl branching formulae

The Young tableaux description $\{l_1, l_2, l_3\}$ of an irrep of $SU(4)$ uses curly brackets and integers such that $l_1 \geqslant l_2 \geqslant l_3 \geqslant 0$. The highest weight notation $(\lambda_1, \lambda_2, \lambda_3)$ uses parentheses and integers $\lambda_1, \lambda_2, \lambda_3$ all $\geqslant 0$. The connection between the two is given by

$$\lambda_1 = l_1 - l_2 \qquad \lambda_2 = l_2 - l_3 \qquad \lambda_3 = l_3. \tag{10}$$

We wish to identify the irreps that are of class one w.r.t. the subgroup $H(2, 2, 1) = SU(2) \otimes SU(2) \otimes U(1)$. For this purpose we shall proceed in two steps each based on an application of the Weyl branching formula [9], last section of chapter 5, or [10], chapter 14 for the branching of an irrep of $SU(n + 1)$ w.r.t. its $SU(n) \times U(1)$ subgroup. First for $n = 3$ [11] we write

$$\chi(\{l_1, l_2, l_3\}, \epsilon) = \sum_{m_1=l_2}^{l_1} \sum_{m_2=l_3}^{l_2} \sum_{m_3=0}^{l_3} \chi(\{m_1 - m_3, m_2 - m_3\}, \eta)\rho^{m-3l/4} \tag{11}$$

where $m = m_1 + m_2 + m_3$, $l = l_1 + l_2 + l_3$. Here $\{m_1 - m_3, m_2 - m_3\}$ denotes an $SU(3)$ irrep in the Young tableaux notation. Also the $SU(4)$ parameters have been written in the form

$$\epsilon = (\rho^{-1/4}\eta, \rho^{-3/4}) \tag{12}$$

appropriate to the restriction from $SU(4)$ to $SU(3) \otimes U(1)$, with η for $SU(3)$ and ρ for corresponding to the $U(1)$ generator

$$Z = \tfrac{1}{4}\mathrm{diag}(1, 1, 1, -3). \tag{13}$$

Writing

$$\eta = (\sigma^{1/3}\tau^{1/2}, \sigma^{1/3}\tau^{-1/2}, \sigma^{-2/3}) \tag{14}$$

we restrict the $SU(3)$ element in (11) to its $SU(2) \otimes U(1)$ subgroup, using parameters which correspond to the $SU(3)$ generators

$$I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{15}$$

Next we use the $SU(3)$ branching formula

$$\chi(\{m_1 - m_3, m_2 - m_3\}, \eta) = \sum_{p_1=m_2}^{m_1} \sum_{p_2=m_3}^{m_2} \chi(p_1 - p_2, \tau)\sigma^{p_1+p_2-2m/3}. \tag{16}$$

Here, with $2I = p_1 - p_2$, the $SU(2)$ character involved is

$$\chi(2I, \tau) = \sum_{k=-I}^{k=+I} \tau^k = \frac{\tau^{I+1/2} - \tau^{-I-1/2}}{\tau^{1/2} - \tau^{-1/2}}. \tag{17}$$

Substitution of (16) into (11) would yield the decomposition appropriate to restriction on the subgroup with generators $\mathbf{I}$, Y, Z of $SU(4)$ represented by

$$I_3 = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \qquad Y = \frac{1}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix} \qquad Z = \frac{1}{4}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}. \tag{18}$$

However, we need a decomposition w.r.t. the subgroup $H(2, 2, 1)$ with generators $\mathbf{I}$, $\mathbf{J}$, U, where $\mathbf{I}$ and $\mathbf{J}$ are the commuting generators of the two $SU(2)$ subgroups of $H(2, 2, 1)$. Thus the matrices we use for our Cartan subalgebra generators are I_3, J_3, U with I_3 given by (18) and

$$J_3 = \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} \qquad U = \frac{1}{4}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{19}$$

Here U is the $U(1)$ generator that commutes with $\mathbf{I}$ and $\mathbf{J}$. To accomodate this point of view, we change our $SU(4)$ parametrization from

$$\epsilon = (\rho^{1/4}\sigma^{1/3}\tau^{1/2}, \rho^{1/4}\sigma^{1/3}\tau^{-1/2}, \rho^{1/4}\sigma^{-2/3}, \rho^{-3/4}) \tag{20}$$

to

$$\epsilon = (\phi^{1/4}\tau^{1/2}, \phi^{1/4}\tau^{-1/2}, \phi^{-1/4}\lambda^{1/2}, \phi^{-1/4}\lambda^{-1/2}) \tag{21}$$

where τ, λ, ϕ are associated evidently with I_3, J_3, U.

We thus finally reach the formula

$$\chi(\{l_1, l_2, l_3\}, \epsilon) = \sum_{m_1=l_2}^{l_1} \sum_{m_2=l_3}^{l_2} \sum_{m_3=0}^{l_3} \sum_{p_1=m_2}^{m_1} \sum_{p_2=m_3}^{m_2} \chi(p_1 - p_2, \tau)\lambda^{m-l/2-p/2}\phi^{(2p-l)/4} \tag{22}$$

where m and l are as above and $p = p_1 + p_2$.

2.2. *Occurrence of the identity irrep of $H(2, 2, 1)$*

Equation (22) allows immediate identification of conditions under which the identity irrep of $SU(2) \otimes SU(2) \otimes U(1)$ with generators $\mathbf{I}$, $\mathbf{J}$, U occurs in the irrep $\{l_1, l_2, l_3\}$ of $SU(4)$. These include

$$p_1 = p_2 \qquad 2m = l + p \qquad l = 2p. \tag{23}$$

Since $p_1 = p_2$ necessarily requires $p_1 = m_2 = p_2$, (23) implies

$$m_2 = \tfrac{1}{4}l. \tag{24}$$

Since m_2 is integral, it follows that we require that l is divisible by 4. Just as an irrep of the first $SU(2)$ needs labels I, I_3, so also the second one needs J, J_3, and J has not yet been brought into the picture. We can achieve this by looking at the portion of (22) that contains all the terms with $I = U = 0$. These are

$$\sum_{m_1=l_2}^{l_1} \sum_{m_3=l_3}^{l_2} \lambda^{m_1+m_3-l/2} = \chi(l_3, \lambda)\chi(l_1 - l_2, \lambda) = \sum_r \chi(2r, \lambda) \tag{25}$$

where, as a result of the $SU(2)$ Clebsch–Gordan series, the upper limit on the sum over r is $\frac{1}{2}(l_1 - l_2 + l_3)$, and the lower is $\frac{1}{2}|l_1 - l_2 - l_3|$. Also $J = 2r$ is the second $SU(2)$ irrep label, and we see that $J = 0$ occurs in (25) exactly once if

$$l_1 - l_2 = l_3. \tag{26}$$

The conditions under which the identity irrep of $H(2, 2, 1)$ occurs in the decomposition of the $SU(4)$ irrep $\{l_1, l_2, l_3\} = (\lambda_1, \lambda_2, \lambda_3)$ w.r.t. $H(2, 2, 1)$ are thus

$$l_1 - l_2 = l_3 \qquad \text{and} \qquad \tfrac{1}{4}(l_1 + l_2 + l_3) \in \mathbb{N}_+. \tag{27}$$

Since the identity irrep occurs once in the decomposition iff these conditions are satisfied, it follows that $H(2, 2, 1)$ is a massive subgroup of $SU(4)$ in the sense of [1]. The conditions (27) imply that the highest weight labels $\lambda_1, \lambda_2, \lambda_3$ of the class one irreps of $SU(4)$ satisfy

$$\lambda_1 = \lambda_3 \qquad \text{and} \qquad \lambda_2 \in \tfrac{1}{2}\mathbb{N}_+ \tag{28}$$

so that we can say that the irreps of $SU(4)$ of class one w.r.t. to $H(2, 2, 1)$ are

$$(r, 2s, r) \qquad \text{where} \qquad r, s \in \mathbb{N}_+. \tag{29}$$

These irreps (see, e.g., [12]) have dimension given by

$$\dim(r, t, r) = \tfrac{1}{12}(1 + t)(1 + r)^2(2 + r + t)^2(2r + t + 3) \tag{30}$$

and eigenvalue

$$c_2(r, t, r) = \tfrac{1}{4}\big(r^2 + rt + \tfrac{1}{2}t^2 + 3r + 2t\big) \tag{31}$$

of the quadratic Casimir operator of $SU(4)$, the normalization having been chosen so that for the adjoint irrep $c_2(1, 0, 1) = 1$. For $t = 2s$, (31) gives, to within an overall multiplicative constant, the energy eigenvalues of the Schrödinger equation of $G(4, 2, \mathbb{C})$, and then (30) gives their degeneracy. Note that (31) implies that $c_2(r, 2s, r) \in \frac{1}{2}\mathbb{N}_+$ for all integers r, s. Table 1 gives data for some low values of r, t.

2.3. *Irreps of* $SU(n + 1)$ *of class one relative to* $H(n - 1, 2, 1)$

The procedure of sections 2.1 and 2.2 can be generalized more or less directly to identifying the irreps of $SU(n + 1)$ of class one relative to $SU(n - 1) \otimes SU(2) \otimes U(1)$. We find that, for $n \geqslant 4$, these are the irreps

$$(r, s, 0^{n-4}, s, r) \qquad r, s \in \mathbb{N}_+ \tag{32}$$

in the highest weight notation. In the $n = 5$ case of $SU(6)$ and $SU(4) \otimes SU(2) \otimes U(1)$, decompositions are given in [8] of about 30 irreps of relatively low dimension. These all conform to the identification (32). One sees there that $(1, 0, 0, 0, 1) = ad = 35$, $(0, 1, 0, 1, 0) = 189$ and $(2, 0, 0, 0, 2) = 405$ are of class one.

The identification (32) enables us, referring to [12], to state e.g. that the spectrum of the Schrödinger equation on $G(6, 2, \mathbb{C})$ involves energy levels given, to within an overall multiplicative constant, by

$$c_2(r, s, 0, s, r) = \tfrac{1}{6}(r^2 + 2rs + 2s^2 + 5r + 8s) \tag{33}$$

of degeneracy, in the absence of accidental cases, given by

$$\begin{aligned}\dim(r, s, 0, s, r) = \frac{1}{2^8 3^3 5}&[(1 + r)(1 + s)(2 + s)(2 + r + s)\\ &\times (3 + r + s)(4 + r + 2s)]^2(3 + 2s)(5 + 2r + 2s).\end{aligned} \tag{34}$$

3. Decomposition of (r, t, r) into irreps of $H(2, 2, 1)$

3.1. Statement of the result

Writing $x = 2I$, $y = 2J$, $z = 2K$, the required decomposition is

$$\chi(r, t, r, \tau, \lambda, \phi) = \sum_{p=0}^{r} \sum_{x=0}^{2p} \sum_{k=0}^{t} \chi(x + k, \tau)\chi(y + k, \lambda)\chi(z + t - k, \phi) \tag{35}$$

where

$$y = 2p - x \qquad z = \min(x, y). \tag{36}$$

In (35), all $\chi(\cdot\,, \tau)$, $\chi(\cdot\,, \lambda)$ are true $SU(2)$ characters, associated with the two $SU(2)$ subgroups of $H(2, 2, 1)$. But $\chi(\cdot\,, \phi)$ is not a true $SU(2)$ character: there is no implication that there is a spin $\mathbf{K}$ such that $K_3 = U$ and such that $\mathbf{K}^2$ and K_3 commute with $\mathbf{I}^2$, I_3, $\mathbf{J}^2$, J_3. Of course K_3, as the $U(1)$ generator, has the required commutation relations; it is $\mathbf{K}^2$ that has not been defined at all. The quantity $\chi(z, \phi)$ arises empirically but demonstrably from the observation that, for each term $\chi(x + k, \tau)\chi(y + k, \lambda)$ of the triple sum in (35), the factors come out in such a way that they can be arranged into (perhaps more than one) sum of the sort

$$\sum_j \phi^j = \chi(z, \phi) \qquad z = 2j_{\max} = -2j_{\min}. \tag{37}$$

The labels z of the sums in question can be seen to be necessary and sufficient to resolve all degeneracies completely. The class one irrep $(1, 2, 1) = 175$ gives a simple illustration. It has $x = y = 1$ or $I = J = \frac{1}{2}$ states with U values $U = -1, 0, 0, 1$ exhibiting degeneracy for $U = 0$. It is natural to group the four states into formal spin type multiplets with $K = 1, 0$, i.e. supposing that linear combinations of the two $U = 0$ states can be assigned to the $K = 1, 0$ multiplets. All sets of U values that arise for given products $\chi(x + k, \tau)\chi(y + k, \lambda)$ in (35) admit such a treatment. The displays in table 2 for various (r, t, r) provide many examples. It is easy by inspection to read the degeneracy of any (x, y, U) state off the displays, and to infer their values for more larger irreps than shown. There is actually much structure to be noted in the displays, appreciation of which plays a part in understanding some aspects of the work described below.

It is a simple matter to check that (35) agrees with (30). The general term of (35) gives a contribution

$$(x + k + 1)(y + k + 1)(z + t + k + 1) \tag{38}$$

to the dimension, and the result follows.

3.2. Proof for $(0, t, 0)$

The basic task in general is to rearrange the five-fold sum in (22) so as to collect λ-powers into true characters $\chi(\cdot\,, \lambda)$, after which the groupings of ϕ-powers into formal characters $\chi(\cdot\,, \phi)$ have to be tackled. The latter simply emerge in a suitable form.

The special case of $(0, t, 0)$, of class one only for even t, is easy. In

$$\chi(0, t, 0) = \sum_{m_2=0}^{t} \sum_{p_1=m_2}^{t} \sum_{p_2=0}^{m_2} \chi(p_1 - p_2, \tau)\lambda^{m_2 - p/2}\phi^{(p-t)/2} \tag{39}$$

we exchange the orders of summations so as to bring the sum on m_2 to the far right of the three summations, and then perform it explicitly. We get

$$\chi(0, t, 0) = \sum_{p_2=0}^{t} \sum_{p_2=0}^{p_1} \chi(p_1 - p_2, \tau)\chi(p_1 - p_2, \lambda)\phi^{(p_1+p_2-t)/2}. \tag{40}$$

The double sum here can straightforwardly be converted into a double sum over $k = p_1 - p_2$ and $c = (p_1 + p_2 - t)/2$, giving

$$\chi(0, t, 0) = \sum_{k=0}^{t} \chi(k, \tau)\chi(k, \lambda) \sum_{c=-(t-k)/2}^{(t-k)/2} \phi^c = \sum_{k=0}^{t} \chi(k, \tau)\chi(k, \lambda)\chi(t - k, \phi). \tag{41}$$

This is in agreement with (35) for $r = 0$, and gives rise to displays like those of table 2A.

3.3. Proof for $(r, 0, r)$

In this case (22) gives

$$\chi(r, 0, r) = \sum_{m_1=r}^{2r} \sum_{m_3=0}^{r} \sum_{p_1=r}^{m_1} \sum_{p_2=m_3}^{r} \chi(p_1 - p_2, \tau)\lambda^{m-2r-p/2}\phi^{(p-2r)/2}. \tag{42}$$

It is necessary to reverse the orders of the first and third, and of the second and fourth sums. This enables each of the sums over m_1 and m_3 to be performed, and leads after minor relabellings to

$$\chi(r, 0, r) = \sum_{p=0}^{r} \sum_{q=0}^{r} \chi(2r - p - q, \tau)\chi(p, \lambda)\chi(q, \lambda)\phi^{(p-q)/2}. \tag{43}$$

Making use of the Clebsch–Gordan series for $SU(2)$, this converts directly into the result

$$\chi(r, 0, r) = \sum_{p=0}^{r} \sum_{x=0}^{2p} \chi(x, \tau)\chi(2p - x, \lambda)\chi(c, \phi) \tag{44}$$

where $c = \min(x, 2p - x)$, also in agreement with (35) for $t = 0$, so that $k = 0$ also. Triangular displays of obvious nature emerge, as illustrated in table 2 of section 5.

3.4. Proof for (r, t, r) *for increasing* t

For $t = 1$ (22) gives

$$\chi(r, 1, r) = \sum_{m_1=r+1}^{2r+1} \sum_{m_2=r}^{r+1} \sum_{m_3=0}^{r} \sum_{p_1=m_2}^{m_1} \sum_{p_2=m_3}^{m_2} \chi(p_1 - p_2, \tau)\lambda^{m-2r-1-p/2}\phi^{(p-2r-1)/2} \tag{45}$$

with m, p, l as before. We treat the contributions arising from the two terms of the sum over m_2 separately. For the $m_2 = r$ contribution, we separate out the contribution A_1 from the $p_1 = m_1$ term of the sum over p_1, leaving a remainder B_1 that is easily calculated by the method of section 3.3 for $(r, 0, r)$. For the $m_2 = r + 1$ contribution, we separate out the contribution A_2 from the $p_2 = m_3$ term of the sum over p_2, leaving a remainder B_2 that is likewise easily calculated by the method of section 3.3 for $(r, 0, r)$. We find

$$B_1 + B_2 = \sum_{p=0}^{r} \sum_{q=0}^{r} \chi(2r - p - q, \tau)\chi(p, \lambda)\chi(q, \lambda)(\phi^{(p-q+1)/2} + \phi^{(p-q-1)/2}). \tag{46}$$

Then, again as in section 3.3, we find

$$B_1 + B_2 = \sum_{p=0}^{r} \sum_{x=0}^{2p} \chi(x, \tau)\chi(2p - x, \lambda)(\chi(c + 1, \phi) + \chi(c - 1, \phi)) \tag{47}$$

where $c = \min(x, 2p - x)$, with twice as many states at each point $(x, 2p - x)$ as (44) yields. Thus, (47) provides the part of the display for any $(r, 1, r)$ which occupies the same triangle as would the full display for $(r, 0, r)$. The contributions A_1 and A_2 fit together easily to give the rest (a single line of entries) of the $(r, 1, r)$ display. We have

$$A_1 + A_2 = \sum_{p=0}^{r}\sum_{q=0}^{r} \chi(2r + 1 - p - q, \tau)\chi(p + q + 1, \lambda)\phi^{(p-q)/2}$$
$$= \sum_{x=0}^{2r} \chi(x + 1, \tau)\chi(2r - x - 1, \lambda)\chi(c, \phi) \qquad (48)$$

where $c = \min(x, 2r - x)$.

If one compares the sum $S = A_1 + A_2 + B_1 + B_2$ and the $t = 1$ case of (35), one sees that they coincide, most easily by comparing each with an appropriate graphical display.

The methods applied to (r, t, r) for $t = 0, 1$ can still fairly easily be employed to prove the $t = 2$ results. There are three terms in the m_2 sums now and the contributions from two specific terms must be extracted therefrom in each case. The three contributions remaining after these subtractions give the part of the display expected to occupy the same triangle in (x, y) plane as would the display for $(r, 0, r)$. The separated terms then are fitted together, naturally and easily, to give the rest of the display. While it is much easier to do the calculation itself than to convey the easily recognizable nature of the path being followed in words, one would at this point expect to prove (35) in general by induction on t. The demonstration of the inductive step is however much too ponderous to be presented here.

The result (35) goes under the heading of branching formula. Reference to the work of mathematicians on this topic can be found in [10, 13]. The case of $G(n+1, 2, \mathbb{C})$ is not treated in these monographs. Some selected references to the large body of papers on branching formulae in theoretical physics are given in section 4.

4. State labelling problem

There is a great deal of good literature addressing the problem of state labelling. General treatment of it is found in [14], which introduces a key concept, that of the *integrity basis*. Referring to the labelling of the states of irreps of a compact Lie group G in a basis adapted to its Lie subgroup H, [14] proves the existence of a finite basis of independent H-scalar operators in the enveloping algebra of G, and explains how this is calculated. Whenever H has a deficit p w.r.t. G, i.e. when H fails by a number p of operators to yield a complete set of commuting operators to use to solve the state labelling problem for the irreps of G, the integrity basis yields $2p$ elements [15], so that any p independent functions of them provides the completion of the required complete set. The deficit 1 example $G = SU(3)$, $H = O(3)$ is given in [14], the methods explained there offer a practical solution of the problem. Following [14] there is an extensive body of work treating other cases of interest. The deficit 2 case of $G = SU(4)$, $H = SU(2) \otimes SU(2)$, with no $U(1)$ factor, is comprehensively analysed in [16], the motivation coming from the Wigner $SU(4)$ spin–isospin multiplet theory. Most of the other works can be traced from [17] or [18]. The latter paper contains discussion of $SU(4)$ irreps, giving results similar in nature to the results in section 2. Our class one irreps (with $H = H(2, 2, 1)$) are in fact degenerate irreps of $SU(4)$ in the sense that this word is used in [18], because $\lambda_1 = \lambda_3 = r$ implies that the cubic Casimir operator of $SU(4)$ vanishes for the class one irreps. Although the methods of [18] could readily be adapted to produce the result, (29), derived in section 2.2, details given in section 2.2 are in any case needed in the discussion of decomposition formulae for class one irreps, and in addition yield the more general result of section 2.3.

The paper [19] presents an integrity basis for the case of interest here, with $G = SU(4)$ and $H = H(2, 2, 1)$. As noted, $H(2, 2, 1)$ provides only five of the six operators that full solution of the state labelling problem needs. The paper cited tells us that we may use any function of two H invariant operators as the sixth member of the $G(4, 2, \mathbb{C})$ complete set. Of these, one is cubic and one is quartic in $SU(4)$ generators. However, the procedures followed are not canonical. They give no clear advice as to how to make an explicit choice of sixth operator in the present example, nor how best to proceed at a similar stage in general. And in fact we wish to choose our sixth operator in a different way.

4.1. The sixth operator for labelling states of irreps of $SU(4)$

In our work, we have used a subset $\mathbf{I}$, I_3, $\mathbf{J}$, J_3, U, of the $SU(4)$ generators as basis for the subgroup $H(2, 2, 1)$. Of the remaining eight generators, four each are positive root and negative root operators. It is easily arranged that the positive ones form a tensor operator

$$T_{m\mu} \qquad m, \mu = \pm\tfrac{1}{2} \tag{49}$$

of rank $1/2$ w.r.t. each of $\mathbf{I}$ and $\mathbf{J}$, with $U = 1/2$. From this one can construct an operator S_+ which is bilinear in the $T_{m\mu}$ and scalar w.r.t. each of $\mathbf{I}$ and $\mathbf{J}$, for which $U = 1$. The negative root operators similarly yield the $U = -1$ scalar $S_- = {S_+}^\dagger$, so that

$$[\mathbf{I}, S_\pm] = 0 \qquad [\mathbf{J}, S_\pm] = 0 \qquad [U, S_\pm] = \pm S_\pm. \tag{50}$$

Thus $S_\pm$ are themselves useful as raising and lowering operators for U. Our intention is to use, as our sixth operator

$$W = [S_+, S_-] \tag{51}$$

which belongs to the identity irrep of $H(2, 2, 1)$. Even though $S_\pm$ function as raising and lowering operators for U, we cannot hope that the commutator in (51) closes on a multiple of U, and thereby enable an $SU(2)$ view of the label K of the previous section to materialize. In fact W is cubic in the generators of $SU(4)$, and must be closely related to the cubic operator(s) used in [19]. We have checked fully that the operator W of (51), appropriately normalized, coincides with the operator UVS–UVT of [19], plus an admisssible tail of terms involving $\mathbf{I}^2$, $\mathbf{J}^2$, U, $\mathcal{C}^{(2)}$, where the last one is the quadratic Casimir of $SU(4)$. Thus use of W as sixth operator is consistent with the general discussion of [19]. It should also be convenient in practice.

5. Tables of $H(2, 2, 1)$ content of some $SU(4)$ irreps

In the following tables, the entries at each point (x, y) give the dimensions of all the $K = \frac{1}{2}z$ multiplets belonging to $x = 2I$, $y = 2J$.

Table 2A. The decompositions of $6 = (0, 1, 0)$, $20 = (0, 2, 0)$ and $50 = (0, 3, 0)$.

3				
2				
1		1		
0	2			
y/x	0	1	2	3

3				
2			1	
1		2		
0	3			
y/x	0	1	2	3

3				1
2			2	
1		3		
0	4			
y/x	0	1	2	3

Table 2B. The decompositions of 15 = ad = (1, 0, 1) and 64 = (1, 1, 1).

3				
2	1			
1		2		
0	1		1	
y/x	0	1	2	3

3		1		
2	2		2	
1		3, 1		1
0	2		2	
y/x	0	1	2	3

Table 2C. The decompositions of 175 = (1, 2, 1) and 384 = (1, 3, 1).

5					
4			1		
3		2		2	
2	3		3, 1		1
1		4, 2		2	
0	3		3		
y/x	0	1	2	3	4

5				1		
4			2		2	
3		3		3, 1		1
2	4		4, 2		2	
1		5, 3		3		
0	4		4			
y/x	0	1	2	3	4	5

Table 2D. The decompositions of 84 = (2, 0, 2) and 300 = (2, 1, 2).

5					
4	1				
3		2			
2	1		3		
1		2		2	
0	1		1		1
y/x	0	1	2	3	4

5		1				
4	2		2			
3		3, 1		3		
2	2		4, 2		2	
1		3, 1		3, 1		1
0	2		2		2	
y/x	0	1	2	3	4	5

Table 2E. The decompositions of 729 = (2, 2, 2) and 1470 = (2, 3, 2).

7							
6			1				
5		2		2			
4	3		3, 1		3		
3		4, 2		4, 2		2	
2	3		5, 3, 1		3, 1		1
1		4, 2		4, 2		2	
0	3		3		3		
y/x	0	1	2	3	4	5	6

7				1				
6			2		2			
5		3		3, 1		3		
4	4		4, 2		4, 2		2	
3		5, 3		5, 3, 1		3, 1		1
2	4		6, 4, 2		4, 2		2	
1		5, 3		5, 3		3		
0	4		4		4			
y/x	0	1	2	3	4	5	6	7

Table 2F. The decompositions of 300 = (3, 0, 3) and 960 = (3, 1, 3).

7							
6	1						
5		2					
4	1		3				
3		2		4			
2	1		3, 1		3		
1		2		2		2	
0	1		1		1		1
y/x	0	1	2	3	4	5	6

7		1						
6	2		2					
5		3, 1		3				
4	2		4, 2		4			
3		3, 1		5, 3		3		
2	2		4, 2		4, 2		2	
1		3, 1		3, 1		3, 1		1
0	2		2		2		2	
y/x	0	1	2	3	4	5	6	7

Table 2G. The decomposition of $2156 = (3, 2, 3)$.

9									
8			1						
7		2		2					
6	3		3, 1		3				
5		4, 2		4, 2		4			
4	3		5, 3, 1		5, 3		3		
3		4, 2		6, 4, 2		4, 2		2	
2	3		5, 3, 1		5, 3, 1		3, 1		1
1		4, 2		4, 2		4, 2		2	
0	3		3		3		3		
y/x	0	1	2	3	4	5	6	7	8

Table 2H. The decomposition of $4096 = (3, 3, 3)$.

9				1						
8			2		2					
7		3		3, 1		3				
6	4		4, 2		4, 2		4			
5		5, 3		5, 3, 1		5, 3		3		
4	4		6, 4, 2		6, 4, 2		4, 2		2	
3		5, 3		7, 5, 3, 1		5, 3, 1		3, 1		1
2	4		6, 4, 2		6, 4, 2		4, 2		2	
1		5, 3		5, 3		5, 3		3		
0	4		4		4		4			
y/x	0	1	2	3	4	5	6	7	8	9

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